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Nonlinear Model Predictive Control for Path Following Problems ⋆

Shuyou Yu *,**, **Xiang Li** **, **Hong Chen** *, **Frank Allgöwer** **

* *Department of Control Science and Engineering, Jilin University, PR China, (e-mail: {shuyou,chenh}@jlu.edu.cn)*
** *Institute for Systems Theory and Automatic Control, University of Stuttgart, Germany, (e-mail: {xiang.li,frank.allgower}@ist.uni-stuttgart.de).*

Abstract: This paper presents a general nonlinear model predictive control (NMPC) scheme for path following problems. Sufficient conditions for recursive feasibility and asymptotic convergence of the given scheme are presented. Furthermore, a method of choosing a suitable terminal penalty and the corresponding terminal constraint is proposed. To illustrate the implementation of the NMPC scheme, the path following problem of a car-like mobile robot is discussed and the control performance is confirmed by simulation results.

Keywords: model predictive control, nonlinear systems, path following problems, car-like mobile robot.

1. INTRODUCTION

Fundamental motion control problems can be roughly classified into three groups, which are point stabilization, trajectory tracking and path following [Soetanto et al., 2003]. The task of the path following problem is to steer a system to follow a reference path not parameterized in time but in its geometrical coordinates. Since the path following problem is capable of involving both point stabilization and trajectory tracking problems, it has received much attention, especially in the robotics field.

By choosing different Lyapunov functions, many nonlinear control approaches to path following problems have been presented [Egerstedt et al., 2001, Li and Zell, 2007, Maček et al., 2005, Ghabcheloo et al., 2005]. However, most of them rarely take the constraints into account, which are crucial for performance and stability [Indiveri et al., 2006, Scolari Conceioçã et al., 2006]. As an effective method of dealing with constraints, nonlinear model predictive control (NMPC) has been applied to the path following problem recently. The papers [Gu and Hu, 2006, Li et al., 2008] show the possibility of implementing NMPC on fast moving wheeled robots, but lack the discussion of choosing the terminal penalty and the terminal constraint. From the theoretical point of view, the papers [Faulwasser et al., 2009, Faulwasser and Findeisen, 2010] suggest an NMPC framework for solving the path following problem, and give sufficient stability conditions. Although some methods of choosing the terminal penalty and the terminal constraint are pointed out, they are either conservative or relying on the system property of differential flatness.

This paper presents a more general NMPC scheme for the path following problems, where the time evolution of the path parameter and its initial value are all determined online. Following a discussion of recursive feasibility and asymptotic convergence, a polytopic linear differential inclusion based method is adopted to choose the terminal penalty and the terminal constraint of the proposed NMPC scheme. As an example, the path following problem of a car-like mobile robot is discussed and solved by the proposed NMPC scheme. The control performance is confirmed by simulations.

The remainder of this paper is organized as follows. Section II sets up the path following problem. In Section III, an NMPC scheme to the path following problem is introduced with the proof of the convergence and feasibility. A method for choosing the terminal penalty and the terminal constraint is presented in detail. Section IV shows the implementation of the proposed NMPC scheme for the path following problem of a car-like mobile robot. A short summary is given in Section V.

2. PATH FOLLOWING PROBLEMS

In the path following problem, the reference path considered is a continuously differentiable geometric curve, which can be defined as a set of points $\mathbf{r}$ parameterized by a scalar s,

$$\mathcal{P} = \{\mathbf{r} \in \mathbb{R}^n |\ \mathbf{r} = \mathbf{p}(s)\}, \tag{1}$$

where the function $\mathbf{p} : \mathbb{R}^1 \to \mathbb{R}^n$ is a twice continuously differentiable function. The scalar s is the curvilinear abscissa with $s \in \mathcal{S} \subseteq \mathbb{R}^1$, where $\mathcal{S}$ is a compact set. The

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value of s is the arc length measured along the path from its origin to the point $\mathbf{p}(s)$.

An intuitive understanding of path following is to approach a reference path as close as possible. When the reference path is defined in (1), path following means to approach a reference point $\mathbf{p}(s(t))$ on the reference path at each time instant t, where t goes from 0 to infinity. The time evolution of $s(t)$ is not necessary to be known *a priori*, but influenced by a virtual input $v(t)$ that is a degree of freedom to choose,

$$\dot{s}(t) = v(t), \quad v \in \mathcal{V} \subseteq \mathbb{R}^1. \tag{2}$$

Consider a continuous time nonlinear system

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{3}$$

with state and input constraints

$$\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n, \quad \mathbf{u} \in \mathcal{U} \subseteq \mathbb{R}^m, \tag{4}$$

where $f(\mathbf{x}, \mathbf{u}) : \mathcal{X} \times \mathcal{U} \longrightarrow \mathbb{R}^n$ is continuously differentiable in $\mathbf{x}$ and $\mathbf{u}$, $\mathcal{U} \subseteq \mathbb{R}^m$ is compact and $\mathcal{X} \subseteq \mathbb{R}^n$ is connected.

The path following problem is:

Given a geometric path $\mathcal{P}$ defined by (1), find admissible control values $\mathbf{u}(t)$ and $v(t)$ such that the error $\mathbf{x}_e(t)$ converges to zero, that is

$$\lim_{t\to\infty} \mathbf{x}_e(t) = 0, \tag{5}$$

where $\mathbf{x}_e$ is defined by

$$\mathbf{x}_e(t) := \mathbf{x}(t) - \mathbf{p}(s(t)). \tag{6}$$

For the reference path $\mathcal{P}$ and the system (3), two technical assumptions are made to guarantee that the reference path can be followed by the system.

***Assumption** 1.* The reference path $\mathcal{P}$ is contained in the state constraint set of system (3), that is $\mathcal{P} \subseteq \mathcal{X}$.

***Assumption** 2.* There exist admissible inputs $\mathbf{u} \in \mathcal{U}$ and $v \in \mathcal{V}$, such that the dynamics of the state $\mathbf{x}(\mathbf{t}) \in \mathcal{X}$ and the parameter $s(t) \in \mathcal{S}$ satisfy

$$\dot{\mathbf{x}}_e(t) = 0, \tag{7}$$

if $\mathbf{x}_e(t) = 0$.

***Remark** 2.1.* Assumption 1 ensures the existence of one $\mathbf{x} \in \mathcal{X}$ matching each point on the reference path $\mathcal{P}$. Together Assumption 1 and Assumption 2 guarantee that the dynamic system can indeed follow the given path.

3. NONLINEAR MODEL PREDICTIVE CONTROL FOR PATH FOLLOWING PROBLEMS

In this section, we will discuss an NMPC scheme for the path following problem. After formulating the online optimization problem, a proof of feasibility and convergence of the introduced NMPC scheme is presented. Furthermore, a polytopic linear differential inclusion (PLDI) based algorithm is proposed to choose the suitable terminal penalty and terminal constraint.

3.1 Optimization Problem and Algorithm

In order to formulate the path following problem within the NMPC framework, we consider the following *online* optimization problem.

***Problem** 1.* For all $\tau \in [t, t+T_p]$,

$$\underset{\mathbf{u}(\cdot,\mathbf{x}(t)),v(\cdot,\mathbf{x}(t)),s(t,\mathbf{x}(t))}{\text{minimize}} \quad J(\mathbf{x}(t)) \tag{8a}$$

subject to

$$\dot{\mathbf{x}}(\tau, x(t)) = f(\mathbf{x}(\tau, x(t)), \mathbf{u}(\tau, x(t))), \tag{8b}$$
$$\dot{s}(\tau, x(t)) = v(\tau, x(t)), \tag{8c}$$
$$\mathbf{x}_e(\tau, x(t)) = \mathbf{x}(\tau, x(t)) - \mathbf{p}(s(\tau, x(t))), \tag{8d}$$
$$\mathbf{u}(\tau, x(t)) \in \mathcal{U}, \ \mathbf{x}(\tau, x(t)) \in \mathcal{X}, \tag{8e}$$
$$v(\tau, x(t)) \in \mathcal{V}, \ s(\tau, x(t)) \in \mathcal{S}, \tag{8f}$$
$$\mathbf{x}_e(t+T_p, \mathbf{x}(t)) \in \Omega, \tag{8g}$$

with

$$J(\mathbf{x}(t)) = E(\mathbf{x}_e(t+T_p, \mathbf{x}(t))) + \int_t^{t+T_p} F(\mathbf{x}_e(\tau, \mathbf{x}(t)), \mathbf{u}_e(\tau, \mathbf{x}(t)))d\tau, \tag{9}$$

where $J(\mathbf{x}(t))$ is the cost functional, and T_p is the prediction horizon.

The terms $E(\mathbf{x}_e(t+T_p, x(t)))$ and $\mathbf{x}_e(t+T_p, \mathbf{x}(t)) \in \Omega$ are the terminal penalty and the terminal constraint, respectively. They are used to guarantee recursive feasibility and achieve asymptotic convergence to the given path. The input value $\mathbf{u}_e(\cdot, \mathbf{x}(t))$ will be discussed in the next subsection. The term $F(\cdot,\cdot)$ is the stage cost function, which specifies the desired control performance and satisfies the following condition.

***Assumption** 3.* $F(\cdot,\cdot) : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^1$ is continuous, and $F(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $F(\mathbf{x}, \mathbf{u}) > 0$ for all $(\mathbf{x}, \mathbf{u}) \in \mathcal{X} \times \mathcal{U} \setminus \{\mathbf{0}, \mathbf{0}\}$.

For clarity, $\mathbf{u}(\tau, \mathbf{x}(t))$ denotes the predicted input function related to the measured state $\mathbf{x}(t)$ at time instant t and $\mathbf{x}(\cdot, \mathbf{x}(t))$ represents the predicted state trajectory starting from $\mathbf{x}(t)$ under the control $\mathbf{u}(\tau, \mathbf{x}(t))$, for all $\tau \in [t, t+T_p]$. The notation $v(\tau, \mathbf{x}(t))$, $s(\tau, \mathbf{x}(t))$ and $\mathbf{x}_e(\tau, \mathbf{x}(t))$ refer to the values of v, s and $\mathbf{x}_e$ at time τ related to $\mathbf{x}(t)$, respectively.

***Remark** 3.1.* The cost functional J and the terminal constraint $\mathbf{x}_e(t+T_p, x(t)) \in \Omega$ do not depend explicitly on the parameter s or v, which is consistent with the fact that s and v only describe a virtual reference motion.

Suppose the sampling time is δ, the proposed NMPC control law is formally described by the following algorithm.

***Algorithm** 1.*

Step 1: Measure system state $\mathbf{x}(t)$ at time t,
Step 2: Find the inputs $s^*(t, \mathbf{x}(t))$, $\mathbf{u}^*(\cdot, \mathbf{x}(t))$ and $v^*(\cdot, \mathbf{x}(t))$ for $\tau \in [t, t+T_p]$ to minimize the value of the cost functional $J(\mathbf{x}(t))$ in Problem 1,
Step 3: Take the input value $\mathbf{u}^*(\tau, \mathbf{x}(t))$, $\tau \in [t, t+\delta]$, as the current input for the system,
Step 4: Take the input value $v^*(\tau, \mathbf{x}(t))$ and the initial state $s^*(t, \mathbf{x}(t))$ to update the path parameter $s(\tau, \mathbf{x}(t))$ for $\tau \in [t, t+\delta]$,
Step 5: Set $t := t+\delta$, go to Step 1.

3.2 Feasibility and Stability

Before the discussion on feasibility and convergence of the proposed NMPC scheme, we denote the error dynamics as

$$\dot{\mathbf{x}}_e := g(\mathbf{x}_e, \mathbf{u}_e), \tag{10}$$

where the control input $\mathbf{u}_e$ is a function of $\mathbf{x}$, s, $\mathbf{u}$ and v.

Due to (1), (2) and (3),

$$\dot{\mathbf{x}}_e = \dot{\mathbf{x}} - [\mathbf{p}(s(t))]' = f(\mathbf{x}, \mathbf{u}) - \frac{\partial \mathbf{p}}{\partial s} v,$$

which shows that the function g depends on the parameters v and s, and is continuous differentiable since $f(\cdot, \cdot)$ is continuous differentiable and $\mathbf{p}(\cdot)$ is twice continuously differentiable.

As a neighborhood of the origin of the error system (10), Ω can be defined as a level set of the positive semi-definite function $E(\cdot)$

$$\Omega := \{\mathbf{x}_e \in \mathbb{R}^n \mid E(\mathbf{x}_e) \leq \alpha\}, \tag{11}$$

with $\alpha > 0$. Besides that, the function g and $\mathbf{u}_e$ have to satisfy the following assumption:

Assumption 4. There exist a compact set $\mathcal{U}_e$ such that $\mathbf{u}_e \in \mathcal{U}_e$ and $0 \in \mathcal{U}_e$. Furthermore, there eixst $s \in \mathcal{S}$ and $v \in \mathcal{V}$ such that $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u} \in \mathcal{U}$ while $\mathbf{x}_e \in \Omega$ and $\mathbf{u}_e \in \mathcal{U}_e$.

As important issues of ensuring feasibility and convergence of the NMPC scheme, the terminal penalty $E(\mathbf{x}_e)$, the terminal set Ω and the corresponding fictitious terminal control law $\pi(\mathbf{x}_e)$ are required to satisfy the following conditions:

B0. $\Omega \subseteq \mathcal{X}$,
B1. $\pi(0) = 0$, and $\pi(\mathbf{x}_e) \in \mathcal{U}_e$, for all $\mathbf{x}_e \in \Omega$,
B2. $E(0) = 0$, and $E(\mathbf{x}_e)$ satisfies

$$\frac{\partial E(\mathbf{x}_e)}{\partial \mathbf{x}_e} g(\mathbf{x}_e, \pi(\mathbf{x}_e)) + F(\mathbf{x}_e, \pi(\mathbf{x}_e)) \leq 0, \tag{12}$$

for all $\mathbf{x}_e \in \Omega$.

Clearly, the terminal set Ω has the following additional properties:

(1) The point $0 \in \mathbb{R}^n$ is contained in the interior of Ω due to the positive definiteness of $E(\mathbf{x}_e)$,
(2) Ω is closed and connected due to the continuity of $E(\mathbf{x}_e)$,
(3) Ω is *robustly* invariant for the nonlinear system (10) controlled by $\mathbf{u}_e = \pi(\mathbf{x}_e)$, for all $s(\cdot) \in \mathcal{S}$ and $v(\cdot) \in \mathcal{V}$ because of (12).

Assumption 5. For the error system (10), there exist a locally asymptotically stabilizing controller $\pi(\mathbf{x}_e)$, a terminal set $\Omega \subseteq \mathcal{X}$ and a continuously differentiable, positive semi-definite function $E(\mathbf{x}_e)$ such that conditions B0-B2 are satisfied for all $\mathbf{x}_e \in \Omega$.

We are ready to show the recursive feasibility of the considered optimization problem and the asymptotic convergence of the path following problem.

Theorem 1. Suppose that

(a) Assumption 1-5 are satisfied,
(b) at the initial time instant, Problem 1 has a feasible solution,

then,

(1) Problem 1 is feasible for all time instants,
(2) the system state $\mathbf{x}(t)$ follows the predefined geometric path $\mathcal{P}$ asymptotically, that is $\lim_{t\to\infty} \mathbf{x}_e(t) = 0$.

Proof: *Assume that Problem 1 has an optimal solution at time instant t, which is $(\mathbf{u}^*(\tau, \mathbf{x}(t)),\ v^*(\tau, \mathbf{x}(t)), s^*(t, \mathbf{x}(t))$ for $\tau \in [t, t+T_p]$. The corresponding input and the state of the error system (10) are $\mathbf{u}_e^*(\tau, \mathbf{x}(t))$, $\mathbf{x}_e^*(\tau, \mathbf{x}(t))$, respectively.*

(1) The input $\mathbf{u}^*(\tau, \mathbf{x}(t))$ is implemented, and the related dynamic of the system (3) is $\mathbf{x}^*(\tau, \mathbf{x}(t))$, for all $\tau \in [t, t+\delta]$. The solution $s^*(\tau, \mathbf{x}(t))$ and $v^*(\tau, \mathbf{x}(t))$, $\tau \in [t, t+T_p]$, are used to get the evolution of the system (2). Since neither model-plant mismatch nor external disturbances are present, $\mathbf{x}(t+\delta) = \mathbf{x}^*(t+\delta, \mathbf{x}(t))$. Thus, the remaining piece of the inputs $\mathbf{u}^*(\tau, \mathbf{x}(t))$ and $v^*(\tau, \mathbf{x}(t))$, $\tau \in [t+\delta, t+T_p]$, satisfies the constraints of Problem 1. Since $\mathbf{x}_e^*(t+T_p, \mathbf{x}(t)) \in \Omega$ and $\mathbf{x}^*(t+T_p) = \mathbf{x}^*(t+T_p, \mathbf{x}(t))$, it follows from Assumption 4 that $\pi(\cdot)$ renders Ω invariant, and there exist $s(\tau, \mathbf{x}^*(t+T_p)) \in \mathcal{S}$ and $v(\tau, \mathbf{x}^*(t+T_p)) \in \mathcal{V}$ such that $\mathbf{x}(\tau, \mathbf{x}^*(t+T_p)) \in \mathcal{X}$ and $\mathbf{u}(\tau, \mathbf{x}^*(t+T_p)) \in \mathcal{U}$, for all $\tau \in [t+T_p, t+T_p+\delta]$. The dynamics of the error system (10) under the terminal control law $\pi(\cdot)$ is $\mathbf{x}_e(\tau, \mathbf{x}^*(t+T_p))$ for all $\tau \geq t+T_p$. Therefore, a feasible solution to Problem 1 at time instant $t+\delta$ is $(\mathbf{u}(\tau, \mathbf{x}(t+\delta)), v(\tau, \mathbf{x}(t+\delta)), s(t+\delta, \mathbf{x}(t+\delta)))$ where $s(t+\delta, \mathbf{x}(t+\delta)) = s^*(t+\delta, \mathbf{x}(t))$, and

$$\mathbf{u}(\tau, \mathbf{x}(t+\delta)) := \begin{cases} \mathbf{u}^*(\tau, \mathbf{x}(t)) & \tau \in [t+\delta, t+T_p), \\ \mathbf{u}(\tau, \mathbf{x}^*(t+T_p)) & \tau \in [t+T_p, t+T_p+\delta], \end{cases}$$

$$v(\tau, \mathbf{x}(t+\delta)) := \begin{cases} v^*(\tau, \mathbf{x}(t)) & \tau \in [t+\delta, t+T_p), \\ v(\tau, \mathbf{x}^*(t+T_p)) & \tau \in [t+T_p, t+T_p+\delta]. \end{cases}$$

(2) Let us define a Lyapunov-like function candidate as

$$V(\mathbf{x}(t)) := \min_{\mathbf{u}^*(\tau, \mathbf{x}(t)), v^*(\tau, \mathbf{x}(t)), s^*(t, \mathbf{x}(t))} J(\mathbf{x}(t)), \tag{13}$$

with $\tau \in [t, t+T_p]$. Note that $0 \leq V(\mathbf{x}(t)) < +\infty$, which follows directly from the definition of $V(\cdot)$ and $V(\mathbf{x}(t)) = 0$ while $\mathbf{x}(t) = \mathbf{p}(s(t))$.
At time instant t, the cost functional is

$$\begin{aligned} V(\mathbf{x}(t)) &= E\left(\mathbf{x}_e^*(t+T_p, \mathbf{x}(t))\right) + \\ &\int_t^{t+T_p} F\left(\mathbf{x}_e^*(\tau, \mathbf{x}(t)), \mathbf{u}_e^*(\tau, \mathbf{x}(t))\right) d\tau. \end{aligned} \tag{14}$$

Considering the feasible solution at time instant $t+\delta$ to Problem 1, and recalled $\pi(\cdot)$ which renders Ω invariant, we have

$$\begin{aligned} &J(\mathbf{x}(t+\delta)) = E\left(\mathbf{x}_e(t+\delta+T_p, \mathbf{x}^*(t+T_p))\right) \\ &+ \int_{t+\delta}^{t+T_p} F\left(\mathbf{x}_e^*(\tau, \mathbf{x}(t)), \mathbf{u}_e^*(\tau, \mathbf{x}(t))\right) d\tau \\ &+ \int_{t+T_p}^{t+T_p+\delta} F\left(\mathbf{x}_e(\tau, \mathbf{x}^*(t+T_p)), \pi(\mathbf{x}_e(\tau, \mathbf{x}^*(t+T_p)))\right) d\tau. \end{aligned} \tag{15}$$

Since the "optimal" solution is better than the feasible solution, we have $V(\mathbf{x}(t+\delta)) \leq J(\mathbf{x}(t+\delta))$. Thus,

$$\begin{aligned} &V(\mathbf{x}(t+\delta)) - V(\mathbf{x}(t)) \leq J(\mathbf{x}(t+\delta)) - V(\mathbf{x}(t)) \\ &= -\int_t^{t+\delta} F\left(\mathbf{x}_e^*(\tau, \mathbf{x}(t)), \mathbf{u}_e^*(\tau, \mathbf{x}(t))\right) d\tau \\ &+ \int_{t+T_p}^{t+T_p+\delta} F(\mathbf{x}_e(\tau, \mathbf{x}^*(t+T_p)), \pi(\mathbf{x}_e(\tau, \mathbf{x}^*(t+T_p)))d\tau \\ &+ E\left(\mathbf{x}_e(t+\delta+T_p, \mathbf{x}^*(t+T_p))\right) - E\left(\mathbf{x}_e^*(t+T_p, \mathbf{x}(t))\right). \end{aligned}$$

Since $E\left(\mathbf{x}_e(t+\delta+T_p, \mathbf{x}^*(t+T_p))\right) - E\left(\mathbf{x}_e^*(t+T_p, \mathbf{x}(t))\right) \leq -\int_{t+T_p}^{t+T_p+\delta} F(\mathbf{x}_e(\tau, \mathbf{x}^*(t+T_p)), \pi(\mathbf{x}_e(\tau, \mathbf{x}^*(t+T_p)))d\tau$ which results from the integration of inequality (12), we have

$$V(\mathbf{x}(t+\delta)) - V(\mathbf{x}(t)) \leq - \int_t^{t+\delta} F(\mathbf{x}_e^*(\tau, \mathbf{x}(t)), \mathbf{u}_e^*(\tau, \mathbf{x}(t)))d\tau.$$

Clearly, $V(\mathbf{x}(t))$ is a monotonically decreasing function and has zero as its low bound. The state of the error system (10) will converge to zero as time increases. Accordingly, the state of the system (3) will finally follow the reference path (1), i.e., $\lim_{t\to\infty} \mathbf{x}_e(t) = 0$. □

3.3 Terminal Set with a Static Terminal Control Law

To choose a suitable pair of the terminal penalty and the terminal constraint that satisfies all the assumptions and conditions above, we will propose a polytopic linear differential inclusion based method. The calculated terminal control law is robust with respect to the parameters v and s, and the terminal set is a related robust invariant set.

Firstly, we discuss how to guarantee the satisfaction of inequality (12) while a quadratic stage cost $F(\mathbf{x}_e, \mathbf{u}_e) := \mathbf{x}_e^T Q \mathbf{x}_e + \mathbf{u}_e^T R \mathbf{u}_e$ is considered.

Suppose that $\mathbf{x}_e = 0$ is an equilibrium of the error system (10), and for all $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u} \in \mathcal{U}$, there exists a matrix

$$[G_x(s,v) \;\; G_u(s,v)] \in \Sigma,$$

such that $g(\mathbf{x}_e, \mathbf{u}_e) = G_x(s,v)\mathbf{x}_e + G_u(s,v)\mathbf{u}_e$. $\Sigma \subseteq \mathbb{R}^n \times \mathbb{R}^{(n+m)}$ is a polytopic linear differential inclusion (PLDI) of the nonlinear system (3) for all $s \in \mathcal{S}$ and $v \in \mathcal{V}$, and is described by its vertices

$$\Sigma := \mathrm{Co}\left\{[A_{1x} \;\; B_{1u}], \ldots, [A_{Nx} \;\; B_{Nu}]\right\}, \tag{16}$$

where $[A_{ix} \;\; B_{iu}]$, $i \in [1, N]$, is the vertex matrix of the set Σ, and N is the number of vertex matries.

Based on the PLDI in (16), a sufficient condition which guarantees the satisfaction of Equation (12) is proposed, that is an inequality subject to a linear matrix inequality (LMI) problem.

Theorem 2. For system (10), suppose that there exist a matrix $X > 0$ and a matrix Y such that

$$\begin{bmatrix} A_{ix}X + B_{iu}Y + (A_{ix}X + B_{iu}Y)^T & X & Y^T \\ X & -Q^{-1} & 0 \\ Y & 0 & -R^{-1} \end{bmatrix} \leq 0, \tag{17}$$

for all $i \in [1, N]$.Then, inequality (12) is satisfied, where $E(\mathbf{x}_e) := \mathbf{x}_e^T P \mathbf{x}_e$, $P = X^{-1}$ and $\pi(\mathbf{x}_e) := Y X^{-1} \mathbf{x}_e$.

***Proof**: For simplicity, denote $K := YX^{-1}$. By substituting $P = X^{-1}$ and $Y = KX$ in (17) and performing a congruence transformation with the matrix $\{X^{-1}, I, I\}$, we obtain*

$$\begin{bmatrix} A_{i,cl}^T P + P A_{i,cl} & X & K^T \\ X & -Q^{-1} & 0 \\ K & 0 & -R^{-1} \end{bmatrix} \leq 0,$$

where $A_{i,cl} := A_{ix} + B_{iu}K$. It follows from the Schur complement that the matrix inequality (17) is sufficient to guarantee

$$\begin{aligned} &[G_x(s,v) + G_u(s,v)K]^T P + \\ &P[G_x(s,v) + G_u(s,v)K] + Q + K^T R K \leq 0. \end{aligned} \tag{18}$$

We choose $E(\mathbf{x_e}) := \mathbf{x_e}^T P \mathbf{x_e}$ as a Lyapunov function candidate, the time derivative of $E(\mathbf{x}_e)$ along the trajectory of (10) is given as follows:

$$\begin{aligned} \frac{dE(\mathbf{x}_e(t))}{dt} =& \dot{\mathbf{x}}_e(t)^T P \mathbf{x}_e(t) + \mathbf{x}_e(t)^T P \dot{\mathbf{x}}_e(t) \\ =& \mathbf{x}_e(t)^T \Big\{ [G_x(s,v) + G_u(s,v)K]^T P + \\ & P[G_x(s,v) + G_u(s,v)K] \Big\} \mathbf{x}_e(t). \end{aligned}$$

Due to (18), we have

$$\frac{dE(\mathbf{x}_e(t))}{dt} \leq -\mathbf{x}_e(t)^T Q \mathbf{x}_e(t) - \mathbf{x}_e(t)^T K^T R K \mathbf{x}_e(t).$$

Thus the inequality (12) holds, and $\pi(\mathbf{x}_e)$ is the associated terminal control law. □

Theorem 2 shows that $\pi(\mathbf{x}_e)$ and $E(\mathbf{x}_e)$ can serve as a candidate for the terminal control law and the terminal penalty for the path following scheme.

From the above discussion, an algorithm is proposed to determine a terminal penalty matrix P and a terminal set Ω *offline* such that inequality (12) holds true and the input constraints $\mathbf{u}_e \in \mathcal{U}_e$ are satisfied.

Algorithm 2.

Step 1: Solve LMI (17) to get a locally stabilizing linear state feedback law $\pi(\mathbf{x}_e)$ and a positive definite matrix P,

Step 2: Find the largest positive α such that $\Omega \in \mathcal{X}$ and $\pi(\mathbf{x}_e) \in \mathcal{U}_e$ for all $\mathbf{x}_e \in \Omega$.

4. PATH FOLLOWING CONTROL OF A CAR-LIKE MOBILE ROBOT

To illustrate the implementation of the proposed NMPC scheme, the path following problem of a car-like mobile robot is considered in this section.

4.1 Problem Formulation

A car-like mobile robot is a kind of nonholonomic robot, which is not able to move in the direction parallel to the wheels' axes. With definition of a world coordinate system $\{W\}$ composed of axes X_w and Y_w shown in Fig. 1, the following equations describe the kinematics model of the car-like mobile robot,

$$\begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\alpha}_R \end{bmatrix} = \begin{bmatrix} v_R \cos\alpha_R \\ v_R \sin\alpha_R \\ \omega_v \end{bmatrix}, \tag{19}$$

where x_R and y_R denote the position of the robot center of mass with respect to $\{W\}$, v_R is the magnitude of the robot translational velocity, α_R denotes the robot moving direction in $\{W\}$ and ω_v is the angular velocity of α_R.

Projecting the robot position and velocity into the path coordinate system $\{M_Q\}$, which is composed of axes X_t and X_n and located at the reference point $\mathbf{M}_Q$, we get the error kinematics model of the path following problem as [Ghabcheloo et al., 2005]

$$\dot{\mathbf{x}}_e = \begin{bmatrix} (y_e c(s) - 1)v + v_R \cos\alpha_e \\ -x_e c(s) v + v_R \sin\alpha_e \\ \omega_v - c(s)v \end{bmatrix}, \tag{20}$$

where the error vector $\mathbf{x}_e = [x_e, y_e, \alpha_e]^T$ is with respect to $\{M_Q\}$, $\alpha_e = \alpha_R - \theta_p$ presents the angular error between the

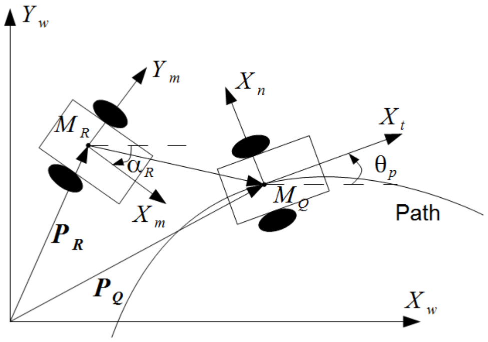

Fig. 1. Path following problem of a car-like mobile robot.

robot moving direction α_R and the path tangent direction θ_p, $c(s)$ denotes the path curvature at point $\mathbf{M}_Q$.

Based on the above setup, the path following problem of a car-like mobile robot can be formulated as follows:

Given a geometric path $\mathcal{P}$ defined by (1), find suitable control laws of v and ω_v to drive the errors x_e, y_e and α_e to zero, while v_R is assigned with a nonzero magnitude value of the desired velocity.

4.2 Simulation Setup

To test the control performance, solving the path following problem of a car-like mobile robot with the proposed NMPC scheme has been simulated by using *Matlab*.

In the simulation, the car-like robot is required to move with a constant velocity $v_R = 0.7$ m/s. The angular velocity is bounded by $-2.5 \le \omega_v \le 2.5$ rad/s. Considering the geometrical symmetry and sharp changes in curvature, an eight-shaped curve is adopted as the reference path depicted as,

$$\begin{aligned} x_P &= 1.8\sin(\psi), \\ y_P &= 1.2\sin(2\psi), \end{aligned} \tag{21}$$

where ψ is a path parameter and determines the path point $[x_P, y_P]$ with respect to the world coordinate system. It has bounded curvature value of $-3.28 \le c(s) \le 3.28$.

4.3 NMPC Controller Design

To implement the proposed NMPC scheme for the path following problem of a car-like mobile robot, the error kinematics model (20) needs to be transformed into the form satisfying $g(0,0) = 0$. Defining states $\mathbf{x}_e = [x_{e1}, x_{e2}, x_{e3}]^T = [x_e, y_e, \alpha_e]^T$ and inputs $\mathbf{u}_e = [u_{e1}, u_{e2}]^T$ with

$$\begin{bmatrix} u_{e1} \\ u_{e2} \end{bmatrix} = \begin{bmatrix} -v + v_R \cos x_{e3} \\ \omega_v - c(s)v \end{bmatrix}, \tag{22}$$

the model (20) can be a candidate of the required error model, where

$$\dot{\mathbf{x}}_e = \begin{bmatrix} x_{e2}c(\mathbf{s})v + u_{e1} \\ -x_{e1}c(\mathbf{s})v + v_R \sin x_{e3} \\ u_{e2} \end{bmatrix}. \tag{23}$$

As the control objective is to drive the states of the error model (23) to zero, a quadratic function is selected as the stage cost function,

$$F(\mathbf{x}_e, \mathbf{u}_e) = \mathbf{x}_e^T Q \mathbf{x}_e + \mathbf{u}_e^T R \mathbf{u}_e, \tag{24}$$

The weight matrices are chosen with $Q = 0.5\mathbf{I}_3$ and $R = 0.5\mathbf{I}_2$, where $\mathbf{I}_j$ denotes the unit diagonal matrix of dimension j. The prediction horizon is 10 and the sampling time δ is 0.02 second.

To choose the terminal penalty and the terminal constraint, we use the scheme presented in Section 3.3. The terminal penalty is

$$E(\mathbf{x}_e(t+T_p)) = \mathbf{x}_e(t+T_p)^T P \mathbf{x}_e(t+T_p). \tag{25}$$

The terminal constraint is chosen as a sub-level set of the terminal penalty, that is, $E(\mathbf{x}_e(t+T_p)) \le \alpha$.

The value of P and α come from Algorithm 2, where the polytopic linear differential inclusion (PLDI) of the error dynamic model (20) is required. According to following partial derivative,

$$\begin{bmatrix} \frac{\partial g}{\partial x_{e1}} & \frac{\partial g}{\partial x_{e2}} & \frac{\partial g}{\partial x_{e3}} & \frac{\partial g}{\partial u_{e1}} & \frac{\partial g}{\partial u_{e2}} \end{bmatrix} = \begin{bmatrix} 0 & c(\mathbf{s})v & 0 & 1 & 0 \\ -c(\mathbf{s})v & 0 & v_R \cos x_{e3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{26}$$

the vertex matrix of the PLDI can be obtained based on the boundary values of v and $c(s)$, while they are all bounded variables defined by a feasible reference path.

Here, by defining $0 \le v \le 1.0$ m/s, $-\frac{\pi}{2} \le \alpha_e \le \frac{\pi}{2}$, and taking the limits of the car-like robot and the eight-shaped reference path into account, we get the following vertex matrices of the PLDI based on (26),

$$[\,A_1\ B_1\,] = \begin{bmatrix} 0 & 3.28 & 0 & 1 & 0 \\ -3.28 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[\,A_2\ B_2\,] = \begin{bmatrix} 0 & -3.28 & 0 & 1 & 0 \\ 3.28 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[\,A_3\ B_3\,] = \begin{bmatrix} 0 & 3.28 & 0 & 1 & 0 \\ -3.28 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[\,A_4\ B_4\,] = \begin{bmatrix} 0 & -3.28 & 0 & 1 & 0 \\ 3.28 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

To satisfy Assumption 4, the range of $\mathbf{u}_e$ is chosen as $|u_{e1}| \le 0.5$ and $|u_{e2}| \le 1.44$. The solution of executing Algorithm 2 results in the terminal penalty parameter $\alpha = 25$ and matrix

$$P = \begin{bmatrix} 28.36 & 0 & 0 \\ 0 & 30.020 & 8.89 \\ 0 & 8.89 & 47.04 \end{bmatrix}.$$

4.4 Comparison Controller

As a benchmark, the well known nonlinear control method showed in [Ghabcheloo et al., 2005] is simulated for comparison. It includes the following equations:

$$\begin{aligned} \sigma(y_e) &= -\mathrm{sgn}(v_R)\sin^{-1}\left(\frac{k_2 y_e}{\|y_e\|+\epsilon_0}\right) \\ \delta(\alpha_e, \sigma) &= \begin{cases} 1 & \textit{if } \alpha_e = \sigma \\ \frac{\sin\alpha_e - \sin\sigma}{\alpha_e - \sigma} & \textit{otherwise} \end{cases} \\ v &= v_R \cos\alpha_e + k_3 x_e \\ \omega_v &= c(s)v + \dot{\sigma} - k_1(\alpha_e - \sigma) - y_e v_R \delta. \end{aligned} \tag{27}$$

For some $k_1,\ k_3 > 0$, $0 < k_2 \le 1$ and $\epsilon_0 > 0$, this controller guarantees global stability, which is proven by choosing the Lyapunov function $V = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2}(\alpha_e - \sigma)^2$. Here the control parameters are chosen as $k_1 = 1$, $k_2 = 0.8$, $k_3 = 20$ and $\epsilon_0 = 30$.

4.5 Simulation Results

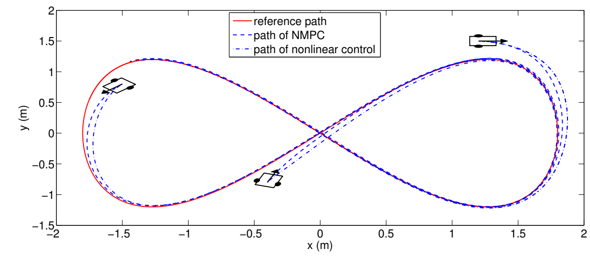

Fig. 2. The reference path and real paths based on the NMPC controller and the nonlinear controller in (27).

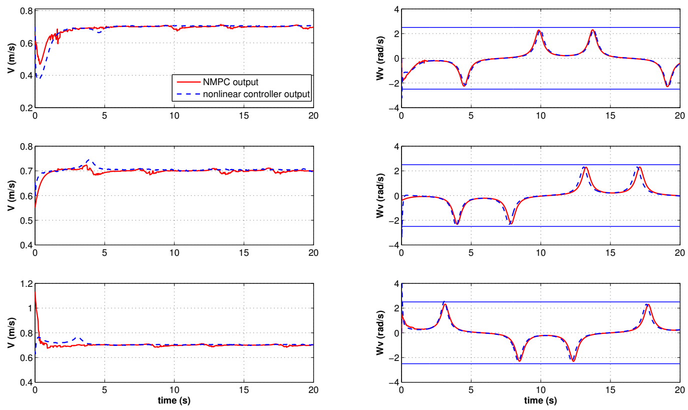

Fig. 3. The velocities v and the angular velocities ω_v from the NMPC controller and the nonlinear controller in (27), which are shown in a solid line and a dashed line, respectively, $t \in [0,\ 20]$.

In the simulation, the robot was started from 3 different initial positions with different heading directions. As Fig. 2 shows, both controllers are capable of driving the robot to follow the reference path, but the path based on the NMPC scheme shows better performance, which means faster convergence. Fig. 3 shows the values of ω_v and v from the two controllers. It is clear that the difference only appear at the initial period of the simulation. Once the robot steps on the reference path, it follows the reference path from then on. Because the nonlinear controller (27) has similar form of the designed u_e in (22), the two controllers have similar control performance when the error $\dot{\mathbf{x}}_e$ is small.

5. CONCLUSIONS

This paper presented a general NMPC scheme for the path following problem, where the time evolution of the path parameter and its initial value are all determined online. Not only the asymptotic convergence and the feasibility of the proposed NMPC scheme, but also a polytopic linear differential inclusion based method to choose the terminal penalty and the terminal constraint were shown. To illustrate the implementation of the proposed NMPC scheme, the path following problem of a car-like mobile robot was discussed in detail. Compared with a well known nonlinear control algorithm, the advantage of the proposed NMPC scheme is shown in the simulation results.

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